Recursive and Non-Recursive Generalized Least-Squares Methods for Estimation of Time Series Models with Exogenous Variables

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Abstract
We present in this paper two generalized least-squares (GLS) methods for estimating regression coefficients of time series models with exogenous variables. The non-recursive GLS method is a generalization of the GLS method suggested by Cochrane and Orcutt (1949). The proposed GLS method consists of a sequence of four linear regressions. A first regression is fitted and provides residuals. These residuals are modeled as an autoregressive process and are used in a second regression (or autoregression) for obtaining estimators of autoregressive coefficients. These estimators are used to generate transformed endogenous and exogenous variables. A third regression makes use of the lagged values of these transformed variables to estimate the regression coefficients. The estimators of the regression coefficients are used to determine the true residuals which are modeled as an ARMA process which is finally used for obtaining the estimators of autoregressive and moving average parameters. The second GLS method is a recursive version of the first GLS method where the estimators are updated at each time point on receipt of the additional observations. The Simulation results based on different model structures with varying numbers of observations are used to compare the performance of our methods with that of exact maximum likelihood (EML) estimates.

Keywords: Recursive and non-recursive estimation, generalized least squares estimation, time series models, multiple-input single-output models, autoregressive moving average disturbances, pseudo linear regression.

1. Introduction
In econometrics and various other fields of applied time series analysis with exogenous variables, the problem of parameters estimation is often concentrated on the regression coefficients which allow giving the importance of economic variables used as exogenous in a regression. The statistical problem dealt with in this paper is the estimation of the vectors \( \beta = (\alpha^T, \lambda^T, \omega_i^T, \delta_i^T, \phi^T, \varrho^T)^T \) of regression coefficients.
There are two main consequences if the ordinary least squares (OLS) method is directly applied for estimating regression models with autocorrelated errors. First, the OLS method shall provide unbiased estimators but which are asymptotically inefficient, because the sampling variances of these estimators may be large compared with those obtained for example by the maximum likelihood (ML) method. Secondly, the application of the OLS method can provide inefficient predictions with needlessly large sampling variances (Johnston, 1960).

One of the ways for overcoming this problem is the use for example the exact maximum likelihood (EML) or the approximate maximum likelihood (AML) methods suggested by Sabiti et al. (2021). Another way is to use the GLS method which is often considered as an ML estimation method when suitable assumptions about the structure of the model are made.

We consider a multiple-input single-output (MISO) model given by the equation

\[
\frac{\alpha(B)}{\lambda(B)} y(t) = \sum_{i=1}^{k} \frac{\omega_i(B)}{\delta_i(B)} x_i(t - b_i) + \frac{\theta(B)}{\phi(B)} \varepsilon(t)
\]

(1)

where \( y(t) \) is the endogenous variable, \( x_i(t) \) \((i = 1, \ldots, k)\) are the exogenous variables, \( B \) is the backshift operator such that \( B^j y(t) = y(t - j) \). \( \varepsilon(t) \) are normally and independently random variables with mean zero and constant variance \( \sigma^2 \), \( b_i \) is the delay of transmission of influence between the \( i \)th exogenous variable and the endogenous variable, or the delay parameter which represents the number of complete time intervals before a change in \( x_i(t) \) begins to have an effect on \( y(t) \) and \( \beta = (\alpha^T, \lambda^T, \omega^T_i, \ldots, \delta^T_i, \phi^T, \theta^T) \) is the vector of parameters where :

\[
\alpha = (\alpha_1, \ldots, \alpha_s)^T, \quad \lambda = (\lambda_1, \ldots, \lambda_r)^T, \quad \omega_i = (\omega_{i0}, \omega_{i1}, \ldots, \omega_{il})^T
\]

(2a)

\[
\delta_i = (\delta_{i1}, \ldots, \delta_{i_l})^T, \quad \phi = (\phi_1, \ldots, \phi_p)^T, \quad \theta = (\theta_1, \ldots, \theta_q)^T
\]

(2b)

and \( l = \bar{s} + \bar{r} + s_1 + \ldots + s_k + r_1 + \ldots + r_k + p + q \) is the number of parameters. The different polynomials in (1) are given by

\[
\alpha(B) = 1 - \sum_{j=1}^{s} \alpha_j B^j, \quad \lambda(B) = 1 - \sum_{j=1}^{r} \lambda_j B^j, \quad \alpha_0 = \lambda_0 = 1
\]

(3a)

\[
\phi(B) = 1 - \sum_{j=1}^{p} \phi_j B^j, \quad \theta(B) = 1 - \sum_{j=1}^{q} \theta_j B^j, \quad \phi_0 = \theta_0 = 1
\]

(3b)

\[
\omega_i(B) = \omega_{i0} + \sum_{j=1}^{l} \omega_{ij} B^j, \quad \delta_i(B) = 1 - \sum_{j=1}^{l} \delta_{ij} B^j, \quad \delta_{i0} = 1, \quad i = 1, \ldots, k.
\]

(3c)

To ensure the stability of the model (1), we need the following assumptions.

Assumption 1: We assume that the roots of the five equations \( \alpha(B) = 0, \quad \lambda(B) = 0, \quad \delta_i(B) = 0 \) \((i = 1,2,\ldots,k)\), \( \phi(B) = 0 \) and \( \theta(B) = 0 \) are outside of the unit circle.

Assumption 2: We also assume that the \( \{\varepsilon(t)\} \) is a strictly stationary martingale difference process satisfying the conditions

\[
E\{\varepsilon(t)|F_{t-1}\} = 0, \quad E\{\varepsilon(t)^2|F_{t-1}\} = \sigma^2, \quad E\{|\varepsilon(t)|^r\} < \infty, \quad r > 2
\]

(4)

Where \( F_t \) is the \( \sigma \)-algebra generated by \( \varepsilon(s) \), \( s \leq t \) and \( \sigma^2 \) is the variance.
Assumption 3. We assume that the vector of input series \( X(t) \) is generated by a \( k \)-dimensional multivariate stochastic process \( X(t) = \{x_1(t), x_2(t), ..., x_k(t)\}^T \) which strictly stationary and ergodic with \( E\{x_i(t)\} < \infty \) \((i = 1, 2, ..., k)\) and \( E\{x_i^2(t)\} < \infty \) \((i = 1, 2, ..., k)\) such that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} X(t) = 0
\]

(5)

\[
\Gamma_X(h) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} X(t)X^T(t + h)
\]

(6)

almost surely (Hannan, 1970 and Hannan and Kavalieris, 1984).

Assumption 4: We assume that the stochastic process generating \( X(t) \) is independent of the stochastic process generating \( \varepsilon(t) \). From that assumption, it follows that

\[
E\{X(t)\varepsilon(t)\} = 0, \quad \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i(t)\varepsilon(t) = 0, \quad i = 1, 2, ..., k.
\]

(7)

2. Pseudolinear regression

We first give an approximation of the MISO model which allows us to obtain a linear regression model in the parameters \( \alpha_j, \lambda_j, \omega_{ij} \) and \( \delta_{ij} \) \((i = 1, ..., k)\). The MISO model (1) can be written in many different linear forms. The first form is to write this model as an infinite lagged regression

\[
\mu(B)y(t) = \sum_{i=1}^{k} \tau_i(B)x_i(t - b_i) + e(t)
\]

(8)

Where the two series \( \mu(B) \) and \( \tau_i(B) \) can respectively be defined as a ratio of two polynomials of finite orders such that

\[
\mu(B) = \frac{\alpha(B)}{\lambda(B)}, \quad \tau_i(B) = \frac{\omega_i(B)}{\delta_i(B)}, \quad i = 1, ..., k
\]

(9)

Where we assume that the coefficients \( \{\alpha_j\} \) and \( \{\tau_{ij}\} \) respectively of the series \( \mu(B) \) and \( \tau_i(B) \) form convergent series, i.e.

\[
\mu(B) = 1 - \sum_{j=1}^{\infty} \mu_j B^j, \quad \sum_{j=1}^{\infty} |\mu_j| < \infty, \quad \mu_0 = 1
\]

(10a)

\[
\tau_i(B) = \tau_{i0} + \sum_{j=1}^{\infty} \tau_{ij} B^j, \quad \sum_{j=1}^{\infty} |\tau_{ij}| < \infty, \quad \tau_{ij} \neq 0.
\]

(10b)

The equation (8) can be approximated by a lagged regression of order \( h \), for sufficiently large \( h \)

\[
y(t) = \sum_{j=1}^{h} \mu_j y(t - j) + \sum_{i=1}^{k} \sum_{j=0}^{h} \tau_{ij} x_i(t - b_i - j) + e(t)
\]

(11)

Where the approximation errors can be made negligible by taking the order \( h \) sufficiently large.
Our procedure for obtaining a linear regression in the parameters $\alpha_j, \lambda_j, \omega_{ij}$ and $\delta_{ij}$ ($i = 1, \ldots, k$) consists to consider the variables $\hat{y}(t)$ and $\hat{x}_i(t)$ such that

$$ \hat{y}(t) = \frac{\alpha(B)}{\lambda(B)} y(t) = \mu(B) y(t) $$

$$ x_i(t) = \frac{\omega_i(B)}{\delta_i(B)} x_i(t) = \tau_i(B) x_i(t) $$

Where $\hat{y}(t)$ and $\hat{x}_i(t)$ will represent respectively the output and the input variables. We can also write (12a) and (12b) respectively as

$$ \hat{y}(t) = y(t) - \sum_{j=1}^{s} \alpha_j y(t - j) + \sum_{j=1}^{r} \lambda_j \hat{y}(t - j) \quad (13a) $$

$$ \hat{x}_i(t) = \sum_{j=0}^{s} \omega_{ij} x_i(t - j) + \sum_{j=1}^{r} \delta_{ij} x_i(t - j) \quad i = 1, \ldots, k. \quad (13b) $$

Thus, the equation (8) can now be written as

$$ y(t) = \sum_{j=1}^{s} \alpha_j y(t - j) - \sum_{j=1}^{r} \lambda_j \hat{y}(t - j) + \sum_{i=1}^{k} \sum_{j=0}^{s} \omega_{ij} x_i(t - b_i - j) $$

$$ + \sum_{i=1}^{k} \sum_{j=1}^{r} \delta_{ij} \hat{x}_i(t - b_i - j) + e(t) \quad (14) $$

Where $e(t)$ is generated by an ARMA $(p, q)$ process given by the equation

$$ e(t) = \frac{\vartheta(B)}{\phi(B)} \epsilon(t) \quad \text{or in the form} \quad \frac{\phi(B)}{\vartheta(B)} e(t) = \epsilon(t) \quad (15) $$

Where $\pi(B)$ forms a convergent series such that

$$ \frac{\phi(B)}{\vartheta(B)} = \pi(B) = 1 - \sum_{j=1}^{\infty} \pi_j B^j, \quad \sum_{j=1}^{\infty} |\pi_j| < \infty, \quad \pi_0 = 1 \quad (16) $$

and the equation (15) can be approximated by an autoregression of order $\hat{h}$

$$ e(t) = \sum_{j=1}^{\hat{h}} \pi_j e(t - j) + \epsilon(t) \quad (17) $$

Where the approximation errors can be made negligible by taking the lag $\hat{h}$ sufficiently large.

If we write the equation (15) as $e(t) = \epsilon(t)/\pi(B)$ and combining by (14) and by the same procedure as in Cochrane and Orcutt (1949), we get that

$$ \pi(B) y(t) = \sum_{j=1}^{s} \alpha_j [\pi(B) y(t - j)] - \sum_{j=1}^{r} \lambda_j [\pi(B) \hat{y}(t - j)] $$

$$ + \sum_{i=1}^{k} \sum_{j=0}^{s} \alpha_{ij} [\pi(B) x_i(t - b_i - j)] + \sum_{i=1}^{k} \sum_{j=1}^{r} \delta_{ij} [\pi(B) \hat{x}_i(t - b_i - j)] + \epsilon(t) \quad (18a) $$
By replacing \( \tilde{y}(t) \) and \( \tilde{x}_i(t) \) given as in (12a-b) in (18), we can also write this equation in the following form

\[
\pi(B)y(t) = \sum_{j=1}^r \alpha_j [\pi(B)y(t-j)] - \sum_{j=1}^r \lambda_j [\Lambda(B)y(t-j)] \\
+ \sum_{i=1}^k \sum_{j=0}^{s_i} \omega_{ij} [\pi(B)x_i(t-b_i-j)] + \sum_{i=1}^k \sum_{j=1}^r \delta_{ij} [\Psi_{ij}x_i(t-b_i-j)] + \varepsilon(t),
\]

(18b)

Since \( \pi(B)\tilde{y}(t) \) and \( \pi(B)\tilde{x}_i(t) \) can be defined respectively as

\[
\pi(B)\tilde{y}(t) = \frac{\alpha(B)\phi(B)}{\lambda(B)\vartheta(B)} y(t) = \Lambda(B)y(t) \\
(19a)
\]

\[
\pi(B)\tilde{x}_i(t) = \frac{\omega(B)\phi(B)}{\delta_i(B)\vartheta(B)} x_i(t) = \Psi_i(B)x_i(t) \\
(19b)
\]

Where

\[
\frac{\alpha(B)\phi(B)}{\lambda(B)\vartheta(B)} = \Lambda(B) = 1 - \sum_{j=1}^\infty \Lambda_j B^j, \quad \sum_{j=1}^\infty |\Lambda_j| < \infty \\
(20a)
\]

\[
\frac{\omega(B)\phi(B)}{\delta_i(B)\vartheta(B)} = \Psi_i(B) = \Psi_{i0} + \sum_{j=1}^\infty \Psi_{ij} B^j, \quad \sum_{j=1}^\infty |\Psi_{ij}| < \infty. \\
(20b)
\]

Then, the equation (18) can now written as

\[
\pi(B)y(t) = \sum_{j=1}^r \alpha_j [\pi(B)y(t-j)] - \sum_{j=1}^r \lambda_j [\Lambda(B)y(t-j)] \\
+ \sum_{i=1}^k \sum_{j=0}^{s_i} \omega_{ij} [\pi(B)x_i(t-b_i-j)] + \sum_{i=1}^k \sum_{j=1}^r \delta_{ij} [\Psi_{ij}x_i(t-b_i-j)] + \varepsilon(t). \\
(21)
\]

If we replace the series

\[
\pi(B) = 1 - \sum_{j=1}^\infty \pi_j B^j, \quad \Lambda(B) = 1 - \sum_{j=1}^\infty \Lambda_j B^j, \quad \Psi_i(B) = \sum_{j=0}^\infty \Psi_{ij} B^j \\
(22a)
\]

Respectively by the polynomials

\[
\pi(B) = 1 - \sum_{j=1}^h \pi_j B^j, \quad \Lambda(B) = 1 - \sum_{j=1}^{h+h} \Lambda_j B^j, \quad \Psi_i(B) = \sum_{j=0}^{h+h} \Psi_{ij} B^j, \\
(22b)
\]

Then the pseudo linear form in the regression coefficients \( \alpha_j, \lambda_j, \omega_{ij} \) and \( \delta_{ij} \) \((i = 1, \ldots, k)\) is given by

\[
\bar{y}(t) = \sum_{j=1}^r \alpha_j \bar{y}(t-j) - \sum_{j=1}^r \lambda_j \bar{y}(t-j) + \sum_{i=1}^k \sum_{j=0}^{s_i} \omega_{ij} \bar{x}_i(t-b_i-j) \\
+ \sum_{i=1}^k \sum_{j=1}^r \delta_{ij} \bar{x}_i(t-b_i-j) + \varepsilon(t) \\
(23)
\]
Where
\[
\ddot{y}(t) = y(t) - \sum_{j=1}^{h} \pi_j \ddot{y}(t-j), \quad \ddot{y}(t) = y(t) - \sum_{j=1}^{h+h} \Lambda_j \ddot{y}(t-j) \tag{24a}
\]
\[
\ddot{x}_i(t) = x_i(t) - \sum_{j=1}^{h} \pi_j x_i(t-j), \quad \ddot{x}_i(t) = \sum_{j=0}^{h+h} \Psi_{ij} x_i(t-j). \tag{24b}
\]
Because \( \epsilon(t) \) in (23) is a white noise, the estimation of this regression will provide strongly consistent estimators of the regression coefficients, under mild assumptions, when the estimators of the coefficients \( \Lambda_j, \Psi_{ij} \) and \( \pi_j \) are known.

If the estimators of the parameters \( \alpha_j, \lambda_j, \omega_{ij} \) and \( \delta_{ij} \) are known, we can evaluate the residuals \( e(t) \) from
\[
e(t) = \alpha(B) \frac{\epsilon(t)}{\lambda(B)} y(t) - \sum_{i=1}^{k} \omega_i(B) \delta_i(B) x_i(t-b_i)
\tag{25}
\]
and the parameters \( \phi_j \) and \( \vartheta_j \) of the ARMA model can be fitted from the equation
\[
e(t) = \sum_{j=1}^{p} \phi_j e(t-j) - \sum_{j=1}^{q} \vartheta_j e(t-j) + \epsilon(t).
\tag{26}
\]

3. The non-recursive GLS method

We assume that the orders \((s, r), (s_1,...,s_k, r_1,...,r_k, b_1,...,b_k)\) and \((p, q)\) are known. The lags \( h \) and \( \bar{h} \) are also assumed to be known or determined by the identification procedure suggested by Sabiti et al. (2021). The proposed GLS method for estimating the parameters \( \alpha_j, \lambda_j, \omega_{ij}, \delta_{ij}, \phi_j \) and \( \vartheta_j \) can be described by the following stages:

**Stage 1**: The regression (11) is estimated by regressing \( y(t) \) on \( y(t-j) \) \( (j=1,...,h) \) and \( x_i(t-b_i-j) \) \( (j=0,1,...,h) \) for obtaining the coefficient estimators \( \hat{\mu}_j \) and \( \hat{\tau}_{ij} \). These estimators are used to estimate the residuals \( \hat{\epsilon}(t) \) as
\[
\hat{\epsilon}(t) = y(t) - \sum_{j=1}^{h} \hat{\mu}_j y(t-j) - \sum_{i=1}^{k} \sum_{j=0}^{h} \hat{\tau}_{ij} x_i(t-b_i-j).
\tag{27}
\]
Put \( J = 0 \) where \( J \) is the iterations counter and proceed with stage 2.

**Stage 2**: The following autoregression of order \( \bar{h} \)
\[
\hat{\epsilon}(t) = \sum_{j=1}^{h} \pi_j \hat{\epsilon}(t-j) + \epsilon(t).
\tag{28a}
\]
is fitted by regressing \( \hat{\epsilon}(t-j) \) \( (j=1,...,\bar{h}) \) for obtaining the coefficient estimators \( \hat{\pi}_j^{(J)} \) and the residuals \( \hat{\epsilon}(t) \) are obtained from
\[
\hat{\epsilon}(t) = \hat{\epsilon}(t) - \sum_{j=1}^{h} \pi_j \hat{\epsilon}(t-j).
\tag{28b}
If \( J = 0 \), put 
\[
\hat{\mu}^{(0)}(B) = \hat{\mu}(B), \quad \hat{\pi}^{(0)}(B) = \hat{\pi}(B), \quad \hat{\tau}^{(0)}_i(B) = \hat{\tau}_i(B), \quad \forall i = 1, \ldots, k. \quad (29)
\]

**Stage 3:** Determine the coefficients \( \hat{\Lambda}^{(J)}_{ij} \) and \( \hat{\Psi}^{(J)}_{ij} \) from
\[
\hat{\Lambda}^{(J)}_{ij}(B) = \hat{\mu}^{(J)}(B)\hat{\pi}^{(J)}_i(B), \quad \hat{\Psi}^{(J)}_{ij}(B) = \hat{\tau}^{(J)}_i(B)\hat{\pi}^{(J)}_i(B) \quad (30a)
\]
if \( J = 0 \) or from
\[
\hat{\lambda}_{ij}^{(J)}(B) = \frac{\hat{\alpha}^{(J)}_{ij}(B)\hat{\phi}^{(J)}_i(B)}{\hat{\lambda}^{(J)}_{ij}(B)\hat{\phi}^{(J)}_i(B)}, \quad \hat{\psi}^{(J)}_{ij}(B) = \frac{\hat{\omega}_{ij}^{(J)}(B)\hat{\phi}^{(J)}_i(B)}{\hat{\psi}^{(J)}_{ij}(B)\hat{\phi}^{(J)}_i(B)} \quad (30b)
\]
if \( J \geq 1 \).

Generate the transformed variables \( \hat{\gamma}(t) \), \( \hat{\xi}_i(t) \) and \( \hat{\xi}_i(t) \) respectively from
\[
\hat{\gamma}(t) = y(t) - \sum_{j=1}^{h} \hat{\pi}_j(t) y(t-j), \quad \hat{\gamma}(t) = y(t) - \sum_{j=1}^{h} \hat{\lambda}_{ij}(t) y(t-j) \quad (31a)
\]
\[
\hat{\xi}_i(t) = x_i(t) - \sum_{j=1}^{h} \hat{\pi}_j^{(J)} x_i(t-j), \quad \hat{\xi}_i(t) = \sum_{j=0}^{h} \hat{\psi}_{ij}^{(J)} x_i(t-j). \quad (31b)
\]

The following regression
\[
\hat{\gamma}(t) = \sum_{j=1}^{\bar{s}} \alpha_j \hat{\gamma}(t-j) - \sum_{j=1}^{\bar{r}} \lambda_j \hat{\gamma}(t-j) + \sum_{i=1}^{k} \sum_{j=0}^{\bar{s}} \omega_{ij} \hat{\xi}_i(t-b_i-j) + \epsilon(t) \quad (32)
\]
is fitted by regressing \( \hat{\gamma}(t) \) on \( \hat{\gamma}(t-j) \) (\( j = 1, \ldots, \bar{s} \)), \( \hat{\gamma}(t-j) \) (\( j = 1, \ldots, \bar{r} \)), \( \hat{\xi}_i(t-b_i-j) \) (\( j = 0,1,\ldots, r_i \)) and \( \hat{\xi}_i(t-b_i-j) \) (\( j = 0,1,\ldots, s_i \)) for obtaining the GLS estimators of \( \alpha_j \), \( \lambda_j \), \( \omega_{ij} \) and \( \delta_{ij} \) denoted respectively by \( \hat{\alpha}_{ij}^{(J+1)} \), \( \hat{\lambda}_{ij}^{(J+1)} \), \( \hat{\omega}_{ij}^{(J+1)} \) and \( \hat{\delta}_{ij}^{(J+1)} \).

**Stage 4:** Determine the residuals \( \hat{e}(t) \) from
\[
e(t) = \frac{\hat{\alpha}^{(J+1)}_{ij}(B)}{\hat{\lambda}^{(J+1)}_{ij}(B)} y(t) - \sum_{i=1}^{k} \frac{\hat{\omega}^{(J+1)}_{ij}(B)}{\hat{\delta}^{(J+1)}_{ij}(B)} x_i(t-b_i) \quad (33a)
\]
The residuals \( \hat{e}(t) \) and \( \hat{e}(t) \) obtained respectively from (28b) and (33) are used in the regression
\[
\hat{\hat{e}}(t) = \sum_{j=1}^{p} \phi_j \hat{\hat{e}}(t-j) - \sum_{j=1}^{q} \delta_j \hat{\hat{e}}(t-j) + \epsilon(t) \quad (33b)
\]
and the ARMA model (33b) is fitted by regressing \( \hat{\hat{e}}(t-j) \) (\( j = 1, \ldots, p \)) and \( \hat{\hat{e}}(t-j) \) (\( j = 1, \ldots, q \)) for obtaining the estimators of the parameters \( \phi_j \) and \( \delta_j \) noted respectively by \( \hat{\phi}^{(J+1)} \) and \( \hat{\delta}^{(J+1)} \). If the following convergence criterion is not reached, increase \( J \) by 1 and repeat stage 2 through 4 where
\[ \max_{i=(1,...,T-p+q)} \left[ \frac{\beta_i^{(J+1)} - \hat{\beta}_i^{(J)}}{\kappa + \beta_i^{(J)}} \right] \geq \gamma, \quad 0 < \kappa, \gamma \leq 1 \] (34)

Where \( \hat{\beta}^T = (\hat{\alpha}^T, \hat{\lambda}^T, \omega_1^T, ..., \omega_k^T, \delta_1^T, ..., \delta_k^T, \phi^T, \hat{\rho}^T) \) and with \( \kappa \) and \( \gamma \) two positive constants sufficiently small. Otherwise, let \( \hat{\beta} = \hat{\beta}^{(J+1)} \) and stop the iterative procedure. Note that for \( J \geq 1 \), the residuals \( \hat{e}(t) \) obtained from (33b) will be used in the estimation of the autoregression (28a) instead of the \( \hat{e}(t) \) obtained at the first stage. The estimators coefficients \( \hat{\Lambda}_j^{(J)} \) and \( \hat{\Psi}_{ij}^{(J)} \) will be obtained from (30b) instead of (30a).

4. The recursive GLS method

The previous GLS method is also called an off-line method in the control literature in which all observations are used simultaneously to find the estimators of the regression coefficients. In the recursive estimation also called on-line method, the vector of regression coefficients are computed recursively in time. See for example Brown, Durbin and Evans (1975) and Söderström and Stoica (1989). This means that if there is an estimator \( \hat{\Phi}_{GLS}(t) \) based on data up to time \( t \), then \( \hat{\Phi}_{GLS}(t+1) \) is computed by some modification of \( \hat{\Phi}_{GLS}(t) \). The recursive methods possess some advantages such that their requirement on primary memory is quite modest, since not all data are stored and they can allow to detect a fault if the system has changed significantly.

If the regression (32) estimated at the third stage of the off-line GLS method is written as
\[
\hat{y}(t) = \hat{X}(t)\Phi + \varepsilon(t) \tag{35}
\]

Where
\[
\hat{X}(t) = \begin{bmatrix}
\hat{y}(t-1), ..., \hat{y}(t-s), \hat{y}(t-1), ..., \hat{y}(t-r), \hat{x}_j(t-b_j), \hat{x}_j(t-b_j-1), \\
..., \hat{x}_j(t-b_j-s_j), \hat{x}_j(t-b_j-1), ..., \hat{x}_j(t-b_j-r_j)
\end{bmatrix}
\]

\[
\Phi = (\alpha^T, \lambda^T, \omega_1^T, ..., \omega_k^T, \delta_1^T, ..., \delta_k^T)^T \tag{36b}
\]

then the estimator of the vector \( \Phi \) is given by
\[
\hat{\Phi}_{GLS}(t) = \left( \sum_{L=1}^{t} \hat{X}(L)\hat{X}^T(L) \right)^{-1} \left( \sum_{L=1}^{t} \hat{X}^T(L)\hat{X}(L) \right) \tag{37}
\]

Where \( \hat{\Phi}_{GLS}(t) \) is the estimator of the vector of regression coefficients based on the first \( t \) observations and the matrix \( \hat{X}(L)\hat{X}^T(L) \) is assumed to be invertible. The argument \( t \) is used to show the dependence of \( \hat{\Phi}_{GLS}(t) \) on time.

We can write \( \hat{\Phi}_{GLS}(t+1) \) under the form
\[
\hat{\Phi}_{GLS}(t+1) = \hat{\Phi}_{GLS}(t) + P(t+1)\hat{X}(t+1) \left( \hat{y}(t+1) - \hat{X}^T(t+1)\hat{\Phi}_{GLS}(t) \right) \tag{38}
\]

Where \( \hat{\Phi}_{GLS}(t+1) \) can be computed from the following formula.
The present algorithm will start at

$$t = L + 1$$

with $$h \geq 1$$. The values of $$b_1, \ldots, b_k$$ are assumed known. We need respectively the initial values of $$\hat{\nu}(0), \hat{\pi}(0), \hat{\Phi}(0)$$ and $$\hat{\Theta}(0)$$. We suggest to take the values of the estimators provided by the off-line GLS method. We also need the covariance matrices $$P_1(0), P_2(0), P_3(0)$$ and $$P_4(0)$$ where we suggest to take $$P_1(0) = D_1, P_2(0) = D_2, P_3(0) = D_3$$ and $$P_4(0) = D_4$$ where $$D_1, D_2, D_3$$ and $$D_4$$ are positive definite matrices provided by the off-line GLS method.

At the initialization, we also need the following vector

$$X_1^T(0) = \begin{bmatrix} y(-1), \ldots, y(-h), x_1(-b_1), x_1(-b_1-1), \ldots, x_1(-b_1-h), \ldots, \\
\ldots, x_k(-b_k), x_k(-b_k-1), \ldots, x_k(-b_k-h) \end{bmatrix}$$

The present algorithm will start at $$t = L + 1$$. The recursive GLS algorithm can be then be summarized by the following stages.

**Stage 1:** The recursive equations for estimating the vector $$\nu^T = \begin{pmatrix} \mu^T, \tau_i^T \end{pmatrix}$$ are respectively given by

$$\hat{\nu}(t+1) = \hat{\nu}(t) + P_1(t)X_1(t+1)\hat{\epsilon}(t+1)$$

(42a)

$$\hat{\epsilon}(t+1) = y(t+1) - X_1^T(t+1)\hat{\nu}(t)$$

(42b)

$$P_1(t+1) = P_1(t) - \frac{P_1(t)X_1(t+1)X_1^T(t+1)P_1(t)}{1 + X_1^T(t+1)P_1(t)X_1(t+1)}$$

(42c)

$$K_1(t+1) = \frac{P_1(t)X_1(t+1)}{1 + X_1^T(t+1)P_1(t)X_1(t+1)}.$$
The recursive equations for estimating the vector \( \pi^T = (\pi_1, \ldots, \pi_h) \) are respectively given by
\[
\hat{\pi}(t + 1) = \hat{\pi}(t) + K_2(t + 1)\hat{\epsilon}(t + 1)
\]
\[
\hat{\epsilon}(t + 1) = \hat{\epsilon}(t + 1) - X_2^T(t + 1)\hat{\pi}(t)
\]
\[
P_2(t + 1) = P_2(t) - \frac{P_2(t)X_2(t + 1)X_2^T(t + 1)P_2(t)}{1 + X_2^T(t + 1)P_2(t)X_2(t + 1)}
\]
\[
K_2(t + 1) = \frac{P_2(t)X_2(t + 1)}{1 + X_2^T(t + 1)P_2(t)X_2(t + 1)}.
\]

Where
\[
X_2^T(t) = \left[ y(t - 1), \ldots, y(t - h), x_1(t - b_1), x_1(t - b_1 - 1), \ldots, x_k(t - b_k), x_k(t - b_k - 1), \ldots, x_k(t - b_k - h) \right]
\]

and \( \hat{\epsilon}(t + 1) \) are also known as recursive residuals.

**Stage 2:** The recursive equations for estimating the vector \( \pi^T = (\pi_1, \ldots, \pi_h) \) are respectively given by
\[
\hat{\nu}(t + 1) = \hat{\nu}(t) + K_2(t + 1)\hat{\epsilon}(t + 1)
\]
\[
\hat{\epsilon}(t + 1) = \hat{\epsilon}(t + 1) - X_2^T(t + 1)\hat{\nu}(t)
\]
\[
P_2(t + 1) = P_2(t) - \frac{P_2(t)X_2(t + 1)X_2^T(t + 1)P_2(t)}{1 + X_2^T(t + 1)P_2(t)X_2(t + 1)}
\]
\[
K_2(t + 1) = \frac{P_2(t)X_2(t + 1)}{1 + X_2^T(t + 1)P_2(t)X_2(t + 1)}.
\]

Where
\[
X_2^T(t) = \left[ \hat{\epsilon}(t - 1), \ldots, \hat{\epsilon}(t - h) \right]
\]

Determine the estimators \( \hat{\Lambda}_j(t + 1) \) and \( \hat{\Psi}_{ij}(t + 1) \) as in (30a), but using the recursive estimators \( \hat{\mu}_j(t + 1) \), \( \hat{\tau}_{ij}(t + 1) \) and \( \hat{\pi}_j(t + 1) \) instead of \( \hat{\mu}_j \), \( \hat{\tau}_{ij} \) and \( \hat{\pi}_j \) respectively. Generate the transformed variables \( \hat{y}(t) \), \( \hat{y}(t) \), \( \hat{x}_j(t) \) and \( \hat{x}_j(t) \) as in (31a-b) using the recursive estimators.

**Stage 3:** The recursive equations for estimating the vector \( \Phi^T = (\alpha_1^T(t), \alpha_k^T(t), \ldots, \alpha_k^T(t), \delta_1^T(t), \ldots, \delta_k^T(t)) \) are respectively given by
\[
\hat{\Phi}_{GLS}(t + 1) = \hat{\Phi}_{GLS}(t) + K_3(t + 1)\hat{\epsilon}(t + 1)
\]
\[
\hat{\epsilon}(t + 1) = \hat{\epsilon}(t + 1) - X_3^T(t + 1)\hat{\Phi}_{GLS}(t)
\]
\[
P_3(t + 1) = P_3(t) - \frac{P_3(t)X_3(t + 1)X_3^T(t + 1)P_3(t)}{1 + X_3^T(t + 1)P_3(t)X_3(t + 1)}
\]
\[
K_3(t + 1) = \frac{P_3(t)X_3(t + 1)}{1 + X_3^T(t + 1)P_3(t)X_3(t + 1)}.
\]

Where
\[
\hat{X}_3^T(t) = \left[ \hat{y}(t - 1), \ldots, \hat{y}(t - s), \hat{y}(t - 1), \ldots, \hat{y}(t - r), \hat{x}_1(t - b_1), \hat{x}_1(t - b_1 - 1), \ldots, \hat{x}_1(t - b_1 - 1), \ldots, \hat{x}_k(t - b_k), \hat{x}_k(t - b_k - 1), \ldots, \hat{x}_k(t - b_k - s_k), \hat{x}_1(t - b_1 - 1), \ldots, \hat{x}_1(t - b_1 - r), \hat{x}_1(t - b_1 - 1), \ldots, \hat{x}_k(t - b_k - r_k) \right]
\]
Stage 4: Determine the residuals $\hat{e}(t)$ from

$$
e(t) = \frac{\hat{\alpha}(B)}{\hat{\lambda}(B)} y(t) - \sum_{i=1}^{k} \hat{\delta}_i(B) x_i(t - b_i)
$$

(45)

Where the estimators of the vector $\Phi^T = (\alpha^T, \lambda^T, \omega_1^T, ..., \omega_k^T, \delta_1^T, ..., \delta_k^T)$ are respectively replacing by the recursive $\hat{\Phi}_{GLS}(t) = \left(\hat{\alpha}^T(t), \hat{\lambda}^T(t), \hat{\omega}_1^T(t), ..., \hat{\omega}_k^T(t), \hat{\delta}_1^T(t), ..., \hat{\delta}_k^T(t)\right)$. The recursive equations for estimating the vector $\Theta^T = (\phi^T, \gamma^T)$ are

$$
\hat{\Theta}(t + 1) = \hat{\Theta}(t) + K_4(t + 1)\hat{e}(t + 1)
$$

(46a)

$$
\hat{e}(t + 1) = \hat{e}(t + 1) - X_4^T(t + 1)\hat{\Theta}(t)
$$

(46b)

$$
P_4(t + 1) = P_4(t) - \frac{P_4(t)X_4(t + 1)X_4^T(t + 1)P_4(t)}{1 + X_4^T(t + 1)P_4(t)X_4(t + 1)}
$$

(46c)

$$
K_4(t + 1) = \frac{P_4(t)X_4(t + 1)}{1 + X_4^T(t + 1)P_4(t)X_4(t + 1)}
$$

(46d)

Where

$$
\hat{X}_4^T(t) = (\hat{e}(t - 1), ..., \hat{e}(t - p), -\hat{e}(t - 1), ..., -\hat{e}(t - q))
$$

(46e)

If an observation is available, the recursive procedure will continue from stage 2. Note that the initial values $\hat{\Phi}_{GLS}(t)$, $\hat{\Theta}(t)$ and $P(t)$ are necessary to start the recursive estimation. As noted in Söderström and Stoica (1989), by the fact that $P(t + 1)$ is the covariance matrix of $\hat{\Phi}_{GLS}(t + 1)$, it is also reasonable to take for $\hat{\Phi}_{GLS}(t)$ an a priori estimator of $\Phi$ and $P(0)$ will reflect the confidence in the initial estimator $\hat{\Phi}_{GLS}(0)$. In this paper, we prefer to take initial values of the estimators those provided by the off-line GLS method. If $P(t)$ is small, then $K(t)$ will be small for all $t$ and the estimators will therefore not change too much from $\hat{\Phi}_{GLS}(t)$. On the other hand, if $P(t)$ is large, the parameter estimators will quickly jump away from $\hat{\Phi}_{GLS}(t)$.

5. Simulation results

To study the performance of the proposed GLS algorithms, we compare it with the exact maximum likelihood (EML). The simulation results that we present are concerned by an ARMAX model where the variables are those studied by Grillenzoni (1990) where $x(t)$ is the exchange rate between the sterling pound and the US dollar, $y(t)$ is the index of wholesale prices in Italy and $t$ = January 1973 – December 1985 (N = 156).

The first simulations concern a MISO model of the form

$$\alpha(B)y(t) = \omega(B)x(t) + \frac{1}{\phi(B)}\varepsilon(t)
$$

(47)
where $\alpha(B) = 1 - \alpha_i B$, $\phi(B) = 1 - \phi_1 B$ and $\omega(B) = \omega_0 + \omega_1 B$. The results given in the following tables are the mean of the estimated parameters, their standard errors and the residual variances.

Table 1: Simulations results for: $b = 0$, $\alpha_i = 0.4$, $\omega_0 = 0.3$, $\omega_1 = 0.02$, $\phi_1 = 0.5$, $N = 40$, 10000 replications

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>$\tilde{\alpha}_i$</th>
<th>$\tilde{\sigma}(\tilde{\alpha}_i)$</th>
<th>$\tilde{\omega}_0$</th>
<th>$\tilde{\sigma}(\tilde{\omega}_0)$</th>
<th>$\tilde{\omega}_1$</th>
<th>$\tilde{\sigma}(\tilde{\omega}_1)$</th>
<th>$\tilde{\phi}_1$</th>
<th>$\tilde{\sigma}(\tilde{\phi}_1)$</th>
<th>$\tilde{\sigma}^2(\tilde{\varepsilon})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GLS</td>
<td>0.346</td>
<td>0.066</td>
<td>0.302</td>
<td>0.010</td>
<td>0.003</td>
<td>0.024</td>
<td>0.565</td>
<td>0.024</td>
<td>1.059</td>
</tr>
<tr>
<td>EML</td>
<td>0.496</td>
<td>0.175</td>
<td>0.307</td>
<td>0.0112</td>
<td>0.093</td>
<td>0.014</td>
<td>0.521</td>
<td>0.182</td>
<td>1.720</td>
</tr>
</tbody>
</table>

Table 2: Simulations results for: $b = 0$, $\alpha_i = 0.4$, $\omega_0 = 0.3$, $\omega_1 = 0.$, $\phi_1 = 0.5$, $N = 80$, 10000 replications

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>$\tilde{\alpha}_i$</th>
<th>$\tilde{\sigma}(\tilde{\alpha}_i)$</th>
<th>$\tilde{\omega}_0$</th>
<th>$\tilde{\sigma}(\tilde{\omega}_0)$</th>
<th>$\tilde{\omega}_1$</th>
<th>$\tilde{\sigma}(\tilde{\omega}_1)$</th>
<th>$\tilde{\phi}_1$</th>
<th>$\tilde{\sigma}(\tilde{\phi}_1)$</th>
<th>$\tilde{\sigma}^2(\tilde{\varepsilon})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GLS</td>
<td>0.436</td>
<td>0.047</td>
<td>0.296</td>
<td>0.008</td>
<td>0.034</td>
<td>0.017</td>
<td>0.649</td>
<td>0.006</td>
<td>1.267</td>
</tr>
<tr>
<td>EML</td>
<td>0.507</td>
<td>0.103</td>
<td>0.297</td>
<td>0.008</td>
<td>0.085</td>
<td>0.009</td>
<td>0.487</td>
<td>0.105</td>
<td>1.340</td>
</tr>
</tbody>
</table>

Table 3: Simulations results for: $b = 0$, $\alpha_i = 0.4$, $\omega_0 = 0.3$, $\omega_1 = 0.02$, $\phi_1 = 0.5$, $N = 120$, 10000 replications

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>$\tilde{\alpha}_i$</th>
<th>$\tilde{\sigma}(\tilde{\alpha}_i)$</th>
<th>$\tilde{\omega}_0$</th>
<th>$\tilde{\sigma}(\tilde{\omega}_0)$</th>
<th>$\tilde{\omega}_1$</th>
<th>$\tilde{\sigma}(\tilde{\omega}_1)$</th>
<th>$\tilde{\phi}_1$</th>
<th>$\tilde{\sigma}(\tilde{\phi}_1)$</th>
<th>$\tilde{\sigma}^2(\tilde{\varepsilon})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GLS</td>
<td>0.376</td>
<td>0.043</td>
<td>0.299</td>
<td>0.005</td>
<td>0.014</td>
<td>0.013</td>
<td>0.618</td>
<td>0.010</td>
<td>1.027</td>
</tr>
<tr>
<td>EML</td>
<td>0.532</td>
<td>0.079</td>
<td>0.292</td>
<td>0.005</td>
<td>0.075</td>
<td>0.005</td>
<td>0.506</td>
<td>0.105</td>
<td>1.502</td>
</tr>
</tbody>
</table>

Table 4: Simulations results for: $b = 0$, $\alpha_i = 0.4$, $\omega_0 = 0.3$, $\omega_1 = 0.02$, $\phi_1 = 0.5$, $N = 156$, 10000 replications

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>$\tilde{\alpha}_i$</th>
<th>$\tilde{\sigma}(\tilde{\alpha}_i)$</th>
<th>$\tilde{\omega}_0$</th>
<th>$\tilde{\sigma}(\tilde{\omega}_0)$</th>
<th>$\tilde{\omega}_1$</th>
<th>$\tilde{\sigma}(\tilde{\omega}_1)$</th>
<th>$\tilde{\phi}_1$</th>
<th>$\tilde{\sigma}(\tilde{\phi}_1)$</th>
<th>$\tilde{\sigma}^2(\tilde{\varepsilon})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GLS</td>
<td>0.389</td>
<td>0.029</td>
<td>0.299</td>
<td>0.003</td>
<td>0.018</td>
<td>0.008</td>
<td>0.557</td>
<td>0.020</td>
<td>1.049</td>
</tr>
<tr>
<td>EML</td>
<td>0.444</td>
<td>0.053</td>
<td>0.287</td>
<td>0.003</td>
<td>0.018</td>
<td>0.003</td>
<td>0.544</td>
<td>0.053</td>
<td>1.030</td>
</tr>
</tbody>
</table>

The second simulations concern the same MISO model of the form

$$\alpha(B) y(t) = \omega(B) x(t) + \frac{1}{\phi(B)} \varepsilon(t)$$  \hspace{1cm} (48)

but where $\alpha(B) = 1 - \alpha_i B$, $\phi(B) = 1 - \phi_1 B - \phi_2 B^2$ and $\omega(B) = \omega_0 + \omega_1 B$.

Table 5: Simulations results for: $b = 0$, $\alpha_i = 0.4$, $\omega_0 = 0.3$, $\omega_1 = 0.02$, $\phi_1 = 0.5$, $\phi_2 = -0.2$, $N = 40$, 10000 replications

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>$\tilde{\alpha}_i$</th>
<th>$\tilde{\sigma}(\tilde{\alpha}_i)$</th>
<th>$\tilde{\omega}_0$</th>
<th>$\tilde{\sigma}(\tilde{\omega}_0)$</th>
<th>$\tilde{\omega}_1$</th>
<th>$\tilde{\sigma}(\tilde{\omega}_1)$</th>
<th>$\tilde{\phi}_1$</th>
<th>$\tilde{\sigma}(\tilde{\phi}_1)$</th>
<th>$\tilde{\phi}_2$</th>
<th>$\tilde{\sigma}(\tilde{\phi}_2)$</th>
<th>$\tilde{\sigma}^2(\tilde{\varepsilon})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GLS</td>
<td>0.353</td>
<td>0.081</td>
<td>0.502</td>
<td>0.012</td>
<td>0.055</td>
<td>0.041</td>
<td>0.840</td>
<td>0.028</td>
<td>-0.442</td>
<td>0.019</td>
<td>1.162</td>
</tr>
<tr>
<td>EML</td>
<td>0.402</td>
<td>0.211</td>
<td>0.511</td>
<td>0.010</td>
<td>0.110</td>
<td>0.016</td>
<td>0.622</td>
<td>0.198</td>
<td>-0.424</td>
<td>0.188</td>
<td>1.872</td>
</tr>
</tbody>
</table>
Table 6: Simulations results for: \( b = 0, \alpha_1 = 0.4, \omega_0 = 0.3, \omega_1 = 0.02, \phi_1 = 0.5, \phi_2 = -0.2, \)
\( N = 80, 10000 \) replications

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>( \tilde{\alpha}_1 )</th>
<th>( \tilde{\sigma}(\tilde{\alpha}_1) )</th>
<th>( \tilde{\omega}_0 )</th>
<th>( \tilde{\sigma}(\tilde{\omega}_0) )</th>
<th>( \tilde{\omega}_1 )</th>
<th>( \tilde{\sigma}(\tilde{\omega}_1) )</th>
<th>( \tilde{\phi}_1 )</th>
<th>( \tilde{\sigma}(\tilde{\phi}_1) )</th>
<th>( \tilde{\phi}_2 )</th>
<th>( \tilde{\sigma}(\tilde{\phi}_2) )</th>
<th>( \tilde{\sigma}_z^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>GLS</td>
<td>0.499</td>
<td>0.040</td>
<td>0.496</td>
<td>0.008</td>
<td>0.133</td>
<td>0.021</td>
<td>0.719</td>
<td>0.017</td>
<td>-0.169</td>
<td>0.011</td>
<td>1.605</td>
</tr>
<tr>
<td>EML</td>
<td>0.234</td>
<td>0.118</td>
<td>0.501</td>
<td>0.009</td>
<td>0.101</td>
<td>0.009</td>
<td>0.776</td>
<td>0.172</td>
<td>-0.372</td>
<td>0.143</td>
<td>1.399</td>
</tr>
</tbody>
</table>

Table 7: Simulations results for: \( b = 0, \alpha_1 = 0.4, \omega_0 = 0.3, \omega_1 = 0.02, \phi_1 = 0.5, \phi_2 = -0.2, \)
\( N = 120, 10000 \) replications

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>( \tilde{\alpha}_1 )</th>
<th>( \tilde{\sigma}(\tilde{\alpha}_1) )</th>
<th>( \tilde{\omega}_0 )</th>
<th>( \tilde{\sigma}(\tilde{\omega}_0) )</th>
<th>( \tilde{\omega}_1 )</th>
<th>( \tilde{\sigma}(\tilde{\omega}_1) )</th>
<th>( \tilde{\phi}_1 )</th>
<th>( \tilde{\sigma}(\tilde{\phi}_1) )</th>
<th>( \tilde{\phi}_2 )</th>
<th>( \tilde{\sigma}(\tilde{\phi}_2) )</th>
<th>( \tilde{\sigma}_z^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>GLS</td>
<td>0.386</td>
<td>0.034</td>
<td>0.500</td>
<td>0.008</td>
<td>0.133</td>
<td>0.021</td>
<td>0.719</td>
<td>0.017</td>
<td>-0.169</td>
<td>0.011</td>
<td>1.055</td>
</tr>
<tr>
<td>EML</td>
<td>0.456</td>
<td>0.058</td>
<td>0.493</td>
<td>0.009</td>
<td>0.101</td>
<td>0.009</td>
<td>0.776</td>
<td>0.173</td>
<td>-0.372</td>
<td>0.143</td>
<td>1.261</td>
</tr>
</tbody>
</table>

Table 8: Simulations results for: \( b = 0, \alpha_1 = 0.4, \omega_0 = 0.3, \omega_1 = 0.02, \phi_1 = 0.5, \phi_2 = -0.2, \)
\( N = 156, 10000 \) replications

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>( \tilde{\alpha}_1 )</th>
<th>( \tilde{\sigma}(\tilde{\alpha}_1) )</th>
<th>( \tilde{\omega}_0 )</th>
<th>( \tilde{\sigma}(\tilde{\omega}_0) )</th>
<th>( \tilde{\omega}_1 )</th>
<th>( \tilde{\sigma}(\tilde{\omega}_1) )</th>
<th>( \tilde{\phi}_1 )</th>
<th>( \tilde{\sigma}(\tilde{\phi}_1) )</th>
<th>( \tilde{\phi}_2 )</th>
<th>( \tilde{\sigma}(\tilde{\phi}_2) )</th>
<th>( \tilde{\sigma}_z^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>GLS</td>
<td>0.395</td>
<td>0.023</td>
<td>0.500</td>
<td>0.003</td>
<td>0.079</td>
<td>0.011</td>
<td>0.476</td>
<td>0.007</td>
<td>-0.055</td>
<td>0.003</td>
<td>1.025</td>
</tr>
<tr>
<td>EML</td>
<td>0.411</td>
<td>0.013</td>
<td>0.485</td>
<td>0.004</td>
<td>0.081</td>
<td>0.004</td>
<td>0.560</td>
<td>0.123</td>
<td>-0.227</td>
<td>0.090</td>
<td>1.164</td>
</tr>
</tbody>
</table>

The different results for the MISO models given in the above tables show that the GLS method provides the values of the parameters which are close to those of the EML method. The use of the optimization algorithm for the EML method gives an advantage for that method when the number of parameters increase.

6. Conclusion
In this paper, we have proposed non-recursive and recursive GLS methods. The non-recursive GLS is a generalization of the method suggested by Cochrane and Orcutt (1949) and consists of a sequence of four linear regressions. The second GLS method is a recursive version of the non-recursive GLS method where the estimators are updated at each time point on receipt of the additional observations. The Simulation results based on different model structures with varying numbers of observations were used to illustrate the performance of the proposed methods.

Conflict of Interest: The authors declare no conflict of interest.

REFERENCES